A Compactness Theorem for Embedded Measured Riemann Surface Laminations

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Abstract

We prove a compactness theorem for embedded measured hyperbolic Riemann surface laminations in a compact almost complex manifold (X, J). To prove compactness result, we show that there is a suitable topology on the space of measured Riemann surface laminations induced by Levy-Prokhorov metric. As an application of the compactness theorem, we show that given a biholomorphism of ϕ of a closed complex manifold X, some power ϕ^k $(k > 0)$ fixes a measured Riemann surface lamination in X.

1 Introduction

Petrovskii-Landis introduced Riemann surface laminations in [LP60] and [PL59] to resolve Hilbert's 16th Problem (see [Hil00]) on the existence and finiteness of limit cycles. Y. Ilyashenko proved Hilbert's conjecture in [Ily60] building on the works of Petrovskii and Landis. Their approach was to study the qualitative (topological) properties of the solutions of the system of differential equation on complex projective space $\mathbb{CP}^2$. Riemann surface laminations appeared in this context naturally.

Riemann surface laminations exhibit many topological properties (See [CGSY03]) while each leaf has an analytic structure. Given a compact almost complex manifold (X, J), one may consider the space of all embedded Riemann surface laminations in X. We denote it by $\mathcal{RSL}(X, J)$. It makes sense to study the global properties of the space $\mathcal{RSL}(X, J)$. Broadly speaking, the study of spaces consisting of geometric structures on the given manifold X or objects embedded in X has yielded very deep results. One may recall the examples of moduli space of J-holomorphic curves in a (fixed) symplectic manifold (M, ω), Teichmuller space, the space of foliations, laminations on a given surface. Very often such spaces admit some geometric structure and nice topology making it easier to study their global properties.

In all instances cited above, compactness or compactification of these spaces plays essential role in establishing results concerning the underlying manifold. For example, in the

Nielsen-Thurston classification theory for surface diffeomorphisms (see [CB88]), the compactness theorem for projective measured laminations on a closed surface Σ with genus atleast two, leads to an interesting result that says any pseudo-Anosov self-diffeomorphism of Σ fixes a lamination. This, further, helps in classifying self-diffeomorphisms of Σ up to free homotopy classes.

Taking inspiration from these examples we ask the following question:

Question 1.1. Is the space $\mathcal{RSL}(X)$ compact with an "appropriate" topology?

One is tempted to consider the Gromov-Hausdorff distance on the space of embedded measured Riemann surface laminations as in the case of the space of projective measured laminations on a surface. However, Gromov-Hausdorff distance takes into account only the distance between the subsets X. It does not necessarily distinguish the lamination structure carried by those sets. A priori, a subset of X could be laminated in two different ways.

It is a useful idea to decorate Riemann surface laminations with measures. We call a Riemann surface lamination endowed with a measure as *measured* Riemann surface lamination. Then we ask the similar question. Is the space of measured Riemann surface lamination, denoted by $\mathcal{MRSL}(X, J)$, compact with a suitable choice of topology? The immediate answer is 'no'. It is essential to have a uniform bound on the measures on Riemann surface laminations without which it is easy see that there may not exist a limit. For instance, take a suitable sequence of J-holomorphic curves (we view them as laminations with transverse measure given by the Dirac delta mass) in X. The uniform bound on measures in case of the J-holomorphic curves, viewed as laminations, is equivalent to giving a uniform bound on the energy of all J-holomorphic curves. In such a case, we know that 'bubbled curves' may appear as a limit, as shown by Gromov in [Gro07]. It is not desirable to have bubbles appear on the leaves of Riemann surface laminations in the limit as bubbles on a particular leaf may intersect other leaves. To prevent the bubbles from appearing in the limit, we need an assumption that the injectivity radius is bounded below uniformly for all Riemann surface laminations.

Moreover, in the case of hyperbolic surfaces, Margulis lemma assured that, the set where injectivity radius is small, is small. Therefore, we have the Deligne-Mumford compactification without any assumption on the injectivity radius. It was shown in [DG] that we do not have any analogue of Margulis lemma for Riemann surface laminations and we cannot expect a compactness result without a lower bound on the injectivity radius. So, one should not even hope for a compactness result for embedded measured Riemann surface laminations without a lower bound on the injectivity radius.

It turns out that we can associate to every element in the space $\mathcal{MSRL}(X, J)$ a unique Borel measures on X. Let $\mathcal{F}$ denote the function that associates a unique Borel measure to a measured Riemann surface laminations. We consider the topology on $\mathcal{MSRL}(X, J)$ induced by the function $\mathcal{F}$ in the following way. The space Borel measures $\mathcal{B}(X)$ on X has Levy-Prokhorov metric (see Definition 2.9). The open sets in the induced topology are pull-back of open sets in $\mathcal{B}(X)$ via the function $\mathcal{F}$. In other words, we consider the coarsest topology $\mathcal{MSRL}(X, J)$ that makes the function $\mathcal{F}$ continuous. Now, our main result can be stated as follows.

Theorem 1.2. *Let $\mathcal{M}(X, J, L, M, \delta) \subset \mathcal{MSRL}(X, J)$ be the space of all embedded L-Lipshitz measured hyperbolic laminations in the compact alomost comlex manifold (X, J) such that the following holds,*

1. *The measure of X under the measure induced by any lamination in $\mathcal{M}(X, J, L, M, \delta)$ is less than or equal to M.*

2. *The injectivity radius of every element in $\mathcal{M}(X, J, L, M, \delta)$ is greater than or equal to δ.*

3. *given $\varepsilon > 0$ there exists $N(\varepsilon)$ such that, if T is a transversal for some lamination in $\mathcal{M}(X, J, L, M, \delta)$, there exists a set $S_{T,\varepsilon}$ such that $|S_{T,\varepsilon}| \leq N(\varepsilon)$ and $m_T(((S_{T,\varepsilon})^{\varepsilon})^C) < \varepsilon$.*

Then, $\mathcal{M}(X, J, L, M, \delta)$ is compact.

Every diffeomorphism $\phi : X \to X$ that preserves J acts on the space J-holomorphic curves in X in a natural way (recall that a J-holomorphic curve in X is a smooth injective map u from compact Riemann surface (Σ, j) into X such that $du \circ j = J \circ du$). Moreover, any J holomorphic curve $\mathcal{L}$ can be viewed as an element of $\mathcal{M}(X, J, L, M, \delta)$ by assigning a finite measure on the transversal which is just a single point.

Using the above compactness theorem we get the following.

Theorem 1.3. *Let (X, J) be a compact almost complex manifold. Let $\phi : X \to X$ be a diffeomorphism that respects the almost complex structure J. Then some (positive) power ϕ^k fixes an embedded measured Riemann surface lamination.*

The case, when almost complex structure J is integrable, is of interest from the complex dynamics viewpoint. As a corollary to the above result, we obtain the following.

Corollary 1.4. *Let X be a compact complex manifold without boundary. Given any biholomophism ϕ from X onto itself, some (positive) power ϕ^k fixes a Riemann surface lamination.*

2 Preliminaries

2.1 Riemann surface lamination

A Riemann surface lamination is a locally compact, separable, metrisable space M with an open cover by flow boxes $\{U_i\}_{i\in I}$ and homeomorphisms $\phi_i : U_i \to D_i \times T_i$, with D_i an open set in $\mathbb{C}$ and T_i a complete separable metric space, such that the coordinate changes in $U_i \cap U_j$ are of the form

$$\phi_j \circ \phi_i^{-1}(z, t) = (\lambda_{ji}(z, t), \tau_{ji}(t))$$

where the map $z \to \lambda_{ji}(z, t)$ is holomorphic for each t.

A map $f : M \to N$ of Riemann surface laminations is holomorphic if it is continuous and maps each leaf of M holomorphically to a leaf of N.

Definition 2.1. An *L-Lipschitz Riemann surface lamination* is a Riemann surface lamination where the coordinate changes $\tau_{ji}(t)$ are L-Lipschitz for all i, j.

2.1.1 Invariant transverse measures

A transverse measure for M is a measure on the σ-ring of transversals which restrict to a σ-finite measure on each transversal and such that each compact regular transversal has finite mass. It is called invariant if it is invariant by the holonomy transformations acting on transversals.

2.2 Measure on a Riemann surface lamination

A Riemann surface lamination does not come equipped with a measure. However, given an invariant transverse measure ν_i on T_i, on each $D_i \times T_i$ there is a natural measure given by

$$\mu(E) = \int_{T_i} \left(\int_{D_i \times \{t\}} \chi_{\phi_i(E)} \cdot \sigma_i \right) d\nu_i(t).$$

where, σ_i is the hyperbolic area measure on D_i.

Let m be the measure on D_i given by $m(A) = \int_{D_i} \chi_A \cdot \sigma_i$

Lemma 2.2. *Suppose W is an open set in $D_i \times T_i$. Let $p \in T_i$, $W_p := \{x | (x, p) \in W\}$ and $\{V_\alpha | \alpha \in A\}$ be a base of open neighbourhoods of p. If $m(W_p) > k$, then there exists $\alpha \in A$ and open set U such that $U \times V_\alpha \subset W$ and $m(U) > k$.*

Proof. Suppose W is open in $D_i \times T_i$ and that $m(W_p) > k$. For each $x \in W_p$, choose an open neighbourhood $U(x)$ of x such that $U(x) \times V_\alpha \in W$ for some $\alpha \in A$. For each $\alpha \in A$, let U_α be the union of those $U(x)$ such that $U(x) \times V_\alpha \in W$. Then, $U_\beta \subset U_\gamma$ whenever $U_\beta \supset U_\gamma$, and $W_p = \cup\{U_\alpha : \alpha \in A\}$. Hence, there is $\alpha \in A$ such that $m(U_\alpha) > k$. Since $U_\alpha \times V_\alpha \subset W$, we are done. □

Lemma 2.3. *μ is a Borel measure.*

Proof. We have, by Lemma 2.2, $f : p \mapsto \mu_0(W_p)$ is lower semicontinuous. That is $f^{-1}((\alpha, \infty))$ is open and hence measurable. As the Borel sigma algebra of $\mathbb{R}$ is generated by sets of the form (α, ∞), f is Borel measurable. Hence, the measure μ is a Borel measure. □

Lemma 2.4. *If $E \subset U_i \cap U_j$ then,*

$$\int_{T_i} \left(\int_{D_i \times \{t\}} \chi_{\phi_i(E)} \cdot \sigma_i \right) d\nu_i(t) = \int_{T_j} \left(\int_{D_j \times \{t\}} \chi_{\phi_j(E)} \cdot \sigma_j \right) d\nu_j(t).$$

Proof. Observe that, $\phi_j \circ \phi_i^{-1}|_{\phi_i(U_i \cap U_j)}$ and hence $\phi_j \circ \phi_i^{-1}|_{\phi_i(E)}$ are biholomorphisms. Thus they are isometries and area-preserving maps, i.e., $(\phi_j \circ \phi_i^{-1})_*(\sigma_i) = \sigma_j$. Also, the transverse measure is holonomy invariant implies that $\nu_j = (\gamma_{ij})_*(\nu_i)$ where γ_{ij} is the holonomy cocycle corresponding to the charts U_i and U_j. Thus, standard chain rule arguments give us the result. □

Definition 2.5. If the lamination structure is given by charts $\phi_i : U_i \to D_i \times T_i$ then, for $E \subset M$ define μ as

$$\mu(E) = \sum_i \int_{T_i} \left(\int_{D_i \times \{t\}} \chi_{\phi_i(E \setminus (\cup_{j=1}^{i-1} U_j)} \cdot \sigma_i \right) d\nu_i(t)$$

Lemma 2.4 tells us that this definition is independent of the ordering of the charts.

2.3 Metric on L-Lipschitz Riemann Surface Laminations

We use the Poincare metric restricted to D_i and the metric on T_i to obtain a metric on the lamination. On $D_i \times T_i$ we define the distance to be the L_1 distance induced by the metrics on D_i and T_i. Then, we define a metric on the lamination using a Kobayashi type construction, namely, we consider the maximal metric on M such that, the maps $\phi_i^{-1} : D_i \times T_i \to U_i$ are distance decreasing functions.

So the distance $d(p, q) = \inf \left\{ \sum_{i=1}^{n} d_{D_{j(i)} \times T_{j(i)}}(\phi_{j(i)}(p_i), \phi_{j(i)}(q_i)) \right\}$ where the pair p_i,q_i belong to the chart $U_{j(i)}$, $p_1 = p$, $q_n = q$ and $q_i = p_{i+1}$. The infimum is taken over all such collection of points $p_1, \ldots, p_n, q_1, \ldots, q_n$ and charts $\left(U_{j(1)}, \phi_{j(1)}\right), \ldots \left(U_{j(n)}, \phi_{j(n)}\right)$. It is easy to see that, this is symmetric and satisfies the triangle inequality. Refer [DG] for a proof that this is a genuine metric.

Definition 2.6 (Injectivity radius for L-Lipschitz hyperbolic Riemann surface laminations)**.** Given an L-Lipschitz Riemann surface lamination $(M, \mathcal{L})$, we say the injectivity radius at a point x is greater than r if there exists an injective map $\varphi : B(0, r)(\subset \mathbb{D}) \times T \to M$ such that:

1. The map $\varphi^{-1}|_{image(\varphi)}$ is a compatible chart.
2. The point $x = \varphi(0, t)$ for some $t \in T$.
3. The map φ is an isometry under the metric defined earlier.
4. The ball $B(x, r) \subset M$ is contained in the image of φ.

2.4 Hyperbolic Riemann surface lamination

A Riemann surface lamination is called a hyperbolic Riemann surface lamination if all leaves are hyperbolic Riemann surfaces. By Theorem 4.3 in [Can93] and Proposition 5.7 in the article on Riemann surface laminations in [CGSY03] we know that this matches with the general definition.

2.5 Measured hyperbolic Riemann surface lamination in X

A measured hyperbolic Riemann surface lamination in a complex manifold X, is a subset A of X with a Riemann surface lamination structure and an invariant transverse measure. Then by Definition 2.5 we have a measure on A. This induces a measure on X, namely, the measure of a set E is the measure of the set $E \cap A$. Thus we can look at the topology induced by the Prohorov metric on Measured Riemann surface laminations in X. We have by Lemma 2.3

Lemma 2.7. *The measure induced by a Riemann surface lamination is a Borel measure.*

Definition 2.8 (Cylindrical chart)**.** A cylindrical chart is a chart $\varphi : \mathbb{D}_\delta \times T \to X$ with a measure m on T and hyperbolic area form σ on D_δ, such that, for all Borel sets $E \subset \mathbb{D}_\delta \times T$,

$$\mu(E) = \int_{T_i} \left(\int_{D_\delta \times \{t\}} \chi_{\phi_i(E)} \cdot \sigma \right) dm(t)$$

Definition 2.9 (Levy-Prokhorov metric)**.** Let $\mathcal{B}(X)$ denote the Borel σ-algebra on X. The Levy-Prokhorov metric d_π between two finite Borel measures μ, ν is defined as

$$d_\pi(\mu, \nu) := \inf \{ \varepsilon > 0 \mid \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \text{ for all } A \in \mathcal{B}(X) \} .$$

where A^ϵ denotes the ϵ-neighborhood of A in X.

Theorem 2.10. *Let (X, d) be a separable metric space and $\mathcal{P}(X)$ the space of all borel measures on X. (X, d) is compact if and only if $(\mathcal{P}(X), d_\pi)$ is compact.*

Proof. Proposition 5.3 in [vG03]. □

2.6 Embedded measured Riemann surface lamination or pseudo-holomorphic lamination

By an embedded measured Riemann surface lamination in a compact complex manifold X, we mean a lamination structure on a closed subset of X. This can be also viewed as a leafwise holomorphic function from a Riemann surface lamination to X. Thus, it will also be called pseudo-holomorphic lamination.

The essential ideas for the proof of Theorem 1.2 are based on arguments presented in [DG]. We modify them to make them suitable to our context. We state the following theorems which appear in [DG] for future use in this article. We will briefly describe those results, but for more details we refer the reader to [DG]. In short, generalising the Gromov-Prokhorov metric, we have defined a distance on the space of distance measure spaces. The distance is defined as follows.

Definition 2.11 (L–isometric embedding)**.** A map $f : (M, d, \mu) \to (N, d', \nu)$ is called an L–isometric embedding if $d'(f(x), f(y)) = d(x, y)$ whenever $\min\{d'(f(x), f(y)), d(x, y)\} < L$.

Definition 2.12 (L–isometric ε–embedding)**.** A map from a distance measure space (M, d, μ) to a distance space (N, d'), $i : (M, d, \mu) \to (N, d')$, is called an L–isometric ε–embedding if there exits $E \subset M$ such that $\mu(E) \leq \varepsilon$ and $i : (M \setminus E) \to N$ is an L–isometric embedding.

Definition 2.13 (Generalised GLP distance, d_ρ)**.** Consider L-isometric ε-embeddings $\iota_i : X_i \to Z, i = 1, 2$ of the spaces X_i into a distance space Z. These give rise to push forward measures $(\iota_i)_*(\mu_i)$ on Z. The distance between the distance measure spaces is the infimum of an appropriate distance between the push forward measures over all L–isometric ε–embeddings, namely,

$$d_\rho = \inf \left\{ d_\pi((\iota_1)_*(\mu_1), (\iota_2)_*(\mu_2)) + \frac{1}{L} + \varepsilon \mid \ \iota_i : X_i \to Z \text{ an } L\text{-isometric } \varepsilon\text{-embedding} \right\},$$

where Z varies over all metric spaces, $L \in \mathbb{R}^+$ and $\varepsilon \in \mathbb{R}^+ \cup \{0\}$.

We have proved that, if two spaces are close to each other with respect to this metric, then there is a quasi-isometry between the two spaces. More precisely,

Theorem 2.14. *Given spaces* (X, d_X, μ) *and* (Y, d_Y, ν) *such that* $d_\rho((X, d_X, \mu), (Y, d_Y, \nu)) < \delta < \frac{1}{\sqrt{2}}$*, there exists a subset* $\widehat{X} \subset X$ *and a function* $f : (X \setminus \widehat{X}) \to Y$ *such that*

- $\widehat{X}$ *has measure less than* δ*,*
- $d_Y(f(x), f(x')) < d_X(x, x') + 2\delta$ *and* $d_X(x, x') < d_Y(f(x), f(x')) + 2\delta$ *for all* x, x' *such that* $d_Y(f(x), f(x')) < \frac{1}{\delta} - 2\delta$ *or* $d_X(x, x') < \frac{1}{\delta} - 2\delta$
- *for all* $E \subset Y$*,* $\mu(f^{-1}(E)) \leq \nu(E^{2\delta}) + 2\delta$ *and* $\nu(E) \leq \mu(f^{-1}(E)^{2\delta}) + 2\delta$*,*
- $\nu(Y) \leq \nu(f(X \setminus \widehat{X})^{2\delta}) + 2\delta$

Further we have,

- $\left(f^{-1}(E)\right)^{2\delta} \subset f^{-1}\left(E^{2\delta}\right)$
- *If* $\delta < \frac{1}{2}$*,* $\nu(E) \leq \mu\left(f^{-1}(E^{2\delta})\right) + 2\delta$*.*

Given maps between sequence of spaces, we showed using this theorem that, we can construct a limit map between the limiting spaces.

Proposition 2.15. *Let* $(X, d_X, \mu), (Y, d_Y, \nu)$ *be complete separable distance measure spaces. Let* $(X^n, d_{X^n}, \mu^n), (Y^n, d_{Y^n}, \nu^n)$ *be such that,* $d_\rho(X^n, X)$ *converges to* 0 *and* $d_\rho(Y^n, Y)$ *converges to* 0*. Given measure preserving isometries* $f^n : X^n \to Y^n$*, there exists a measure preserving isometry* $f : X \setminus \underline{X} \to Y$ *where* $\underline{X} \subset X$ *is a set of measure zero.*

Finally, we have proved a pre-compactness theorem, for the space of distance measurable spaces equipped with the metric we have defined.

Theorem 2.16. *Let* $\mathfrak{X}_{\varepsilon,M}$ *be a collection of complete separable distance measure spaces with the property that* $\mu(X) \leq M$ *for all* $X \in \mathfrak{X}_{\varepsilon,M}$*. Suppose, given* $\varepsilon > 0$ *there exists* $N(\varepsilon)$ *such that, for all* $(X, d, \mu) \in \mathfrak{X}_{\varepsilon,M}$ *there exists a set* $S_{X,\,\varepsilon}$ *such that* $|S_{X,\,\varepsilon}| \leq N(\varepsilon)$ *and* $\mu((S_{X,\,\varepsilon})^{\varepsilon})^C) \leq \varepsilon$*. Then,* $\mathfrak{X}_{\varepsilon,\mathcal{M}}$ *is totally bounded and hence pre-compact.*

3 Borel Measures and Riemann Surface Laminations

In this section we will see two important properties of the measure induced by a measured Riemann surface lamination. Before that, we prove a useful lemma as follows.

Lemma 3.1. *Let* S *be a hyperbolic surface (not necessarily compact or finite genus). Let* A *be a subset of* S *with the hyperbolic area of* A*, denoted by* $\mu(A)$*, nonzero. Then the* 2*-dimensional Hausdorff content* $C_H^2(A)$ *is nontrivial.*

Proof. Assume contrary that $C_H^2(A) = 0$. By definition of the Hausdorff content, for any $\delta > 0$, there is a cover C consisting of balls B_{r_i} with radii r_i such that $\sum r_i^2 < \delta$. This implies that for each i, we have $r_i < \delta$.

Now Bishop-Cheeger-Gromov volume comparison Theorem (see Chapter 9, Lemma 36 from Petersen) implies, for balls with small enough radii, the hyperbolic area $\mu(B_r)$ is comparable with πr^2. More precisely, for r very small, we have

$$\frac{1}{2} < \frac{\pi r^2}{\mu(B_r)}$$

This implies that $\mu(B_r) < 2\pi r^2$.

Applying the above estimate for the cover C we see that

$$\sum \mu(B_{r_i}) < 2\pi \sum r_i^2 < 2\pi\delta$$

On the other hand $\mu(A) \leq \sum \mu(B_{r_i})$ as hyperbolic area is a measure. Thus, $\mu(A) < 2\pi\delta$ for arbitrarily small δ. This implies that $\mu(A) = 0$. Hence a contradiction.

□

Lemma 3.2. *If μ is a measure induced by a Riemann surface laminations in X and E is a set with $\mu(E) \neq 0$, then the 2-dimensional Hausdorff content of E is greater than 0. This property will be called property 1(P-1).*

Proof. Let $\mathcal{L}$ denote the Riemann surface lamination. By definition of induced measure on W, we have $\mu(E) = \mu(E \cap \mathcal{L})$. Since $\mu(E) \neq 0$, there some leaf L of the lamination $\mathcal{L}$ such that $\mu(E \cap L) \neq 0$. By Lemma 3.1, we see that the $C_H^2(E \cap L) \neq 0$. Hence, $C_H^2(E) \neq 0$ □

Lemma 3.3. *Given a point $x \in X$, there exists a cylindrical chart around x. (This will be referred to as property 2 or P-2).*

Proof. If the point x does not belong to the Riemann surface lamination $\mathcal{L}$ then there is an open set B containing x such that $\mathcal{L} \cap B = \emptyset$. As W is a complex manifold, there exist a chart $\psi : \mathbb{C}^n \to W$ around x. Let $\overline{0}$ be the zero vector in $\mathbb{C}^{n-1}$. Define, $\varphi : \mathbb{C} \times \{pt\} \to W$ as $\varphi(x, pt) = \psi(x, \overline{0})$. Observe that $\mu(\mathbb{D}_\delta \times \{pt\}) = 0$ and we take the trivial measure $m(pt) = 0$. Thus, the conclusion holds.

Now assume that $x \in \mathcal{L}$. Then, by the definition of measure on the lamination, we have a cylindrical chart. □

Combining the above two lemmas, we get the following theorem.

Theorem 3.4. *Let μ be the induced Borel measure by the measured Riemann surface lamination $\mathbb{L}$. Then μ satifies the following two properties.*

(A) If E is measurable subset of X with $\mu(E) \neq 0$ then 2-dimensional Hausdorff content of E is non-trivial.

(B) For any point $x \in X$, there is a cylindrical chart around x.

4 Proof Of Compactness theorem

Proof Of Theorem 1.2. Note that $(\mathcal{B}(X), d_\pi)$ is compact as X is compact. As the topology is given by the induced measures, we will show $\mathcal{M}(X, L, M, \delta)$ is a closed under the induced topology. Let $\mathcal{L}_n$ be a sequence in $\mathcal{M}(X, L, M, \delta)$ and μ_n the induced measures. Further let μ_n converge to a Borel measure μ. We will construct a measured Riemann surface lamination $\mathcal{L}$ which induces the measure μ. To this end, we construct an atlas for $\mathcal{L}$ using the atlases for $\mathcal{L}_n$.

Let $x \in Supp(\mu)$ and $B_r(x)$ be the ball of radius r centred at x. As μ_n converges to μ, $d_X(x, supp(\mu_n))$ converges to zero as n goes to infinity. Therefore, there exist a sequence of points $x_n \in Supp(\mu_n)$ which converge to x. So, for all n large enough, $x_n \in B_r(x)$. As $x_n \in Supp(\mu_n)$ and μ_n satisfies P-2, we can find cylindrical charts $\varphi_n : D_{\delta_n} \times T_n \to X$ around x_n. Moreover, as the injectivity radius of $\mathcal{L}_n$ is greater than or equal to r for all n, we can assume, without loss of generality, that $\delta_n \geq \delta$. By abuse of notation, by φ_n we mean $\varphi_n|_{D_\delta \times T_n} : D_\delta \times T_n \to B_r(x)$.

Lemma 4.1. $m_n(T_n) \leq \frac{M}{Area(\mathbb{D}_\delta)}$

Proof. Note that $Area(\mathbb{D}_\delta) \times m_n(T_n) = \mu_n(\mathrm{Im}(\phi_n)) \leq \mu_n(X) \leq M$. Thus, $m_n(T_n) \leq \frac{M}{Area(\mathbb{D}_\delta)}$. □

Thus, by Theorem 2.16 there exist a subsequence $(T_{n_k}, d_{T_{n_k}}, m_{n_k})$ which converges to some metric measure space (T, d, m). We take this subsequence and to simplify the notation assume that (T_n, d_{T_n}, m_n) converges to (T, d, m). We will construct a map $\varphi : D_\delta \times T \to B_r(x)$ as a limit of the maps $\varphi_n : D_\delta \times T_n \to B_r(x)$. More precisely, we construct maps $\psi_n : \left(T \setminus \widehat{T^n}\right) \to T_n$ as in Theorem 2.14 and take the limit of $\varphi_n(z, \psi_n(t))$ as n tends to infinity. We will prove that such a limit exists.

As $d_\rho(T_n, T)$ converges to 0, without loss of generality, we can assume that $\delta_n = d_\rho(T_n, T) < \frac{1}{n^2}$. Let $T(k) = \cup_{i=k}^{\infty} \widehat{T^i}$ and $\underline{T} = \left[\cap_{k=1}^{\infty} T(k)\right]$. Note that, $m(T(k)) \leq \sum_{i=k}^{\infty} \frac{1}{i^2}$ which goes to zero as k goes to infinity. Thus, the measure $m\left(\cap_{k=1}^{\infty} T(k)\right) = 0$.

Lemma 4.2. *Given a point $(z, t) \in D_\delta \times (T \setminus \underline{T})$, the infinite sequence $\varphi_n(z, \psi_n(t))$ has a convergent subsequence.*

Proof. Assume the contrary. Then there exists an ε_1 such that

$$d_Y\left(\varphi_n \circ \psi_n(x), \varphi_m \circ \psi_m(x)\right) \geq \varepsilon_1.$$

Choose $\varepsilon < \min\left(\left\{\left(\frac{1}{\delta_n} - 2\delta_n\right), \left(\frac{\varepsilon_1}{L} - 2\delta_n\right) : n \in \mathbb{N}\right\} \cap \{x \in \mathbb{R} : x > 0\}\right)$. As n goes to infinity, δ_n goes to zero and the terms $\left(\frac{1}{\delta_n} - 2\delta_n\right), \left(\frac{\varepsilon_1}{L} - 2\delta_n\right)$ increase. Hence the minimum exists. By 2.14, we have for all n such that all the terms $\left(\frac{1}{\delta_n} - 2\delta_n\right), \left(\frac{\varepsilon_1}{L} - 2\delta_n\right)$ are positive (i.e for all but finitely many n),

$$\psi_n(B(x, \varepsilon)) \subset B(\psi_n(x), \varepsilon + 2\delta_n) = (B(\psi_n(x), \varepsilon))^{2\delta_n}.$$

Thus,

$$\begin{aligned}\mu\left(\varphi_n\left((B(\psi_n(x),\varepsilon))^{2\delta_n}\right)\right) &= \mu_{\mathbb{D}_\delta\times T_n}((B(\psi_n(x),\varepsilon))^{2\delta_n}) \\ &\geq \mu_{\mathbb{D}_\delta\times T}(B(x,\varepsilon)) - 2\delta_n \\ &\quad \text{(apply Theorem 2.14 twice).}\end{aligned}$$

Furthermore, $\varphi_n\left((B(\psi_n(x),\varepsilon))^{2\delta_n}\right)$ are disjoint. This contradicts with the assumption that μ is a finite measure. □

Let $f_n(z,t) := \varphi_n(z,\psi_n(t))$. Fix a $z\in\mathbb{D}_\delta$ and choose a countable dense subset $S = \{(z_1,t_1),(z_2,t_2),...\}\subset\mathbb{D}_\delta\times(T\setminus\underline{T})$. Choose a subsequence $f_{n_{1,k}}$ of f_n such that $f_{n_{1,k}}(z_1,t_1)$ converges. We choose a further subsequence $f_{n_{2,k}}$ of $f_{n_{1,k}}$ such that $f_{n_{2,k}}(z_2,t_2)$ converges. Observe that $f_{n_{1,k}}(z_1,t_1)$ continues to converge. Iterating this process, we obtain subsequences $f_{n_{j,k}}$ so that $f_{n_{j,k}}(z_l,t_l)$ converges for $l\leq j$. It follows that for the diagonal sequence $f_{n_{k,k}}$ we have the corresponding convergence for all points in S. Replace f_n by this diagonal subsequence. Define

$$\varphi(z_i,t_i) = \lim_{n\to\infty} f_n(z_i,t_i).$$

Lemma 4.3. $d_X(\varphi(z_i,t_i),\varphi(z_j,t_j))\leq d_{\mathbb{D}_\delta}(z_i,z_j) + d_{T\setminus\underline{T}}(t_i,t_j)$

Proof. The metric induced by $\mathcal{L}_n$ induces the subspace topology. So the metric induced by the lamination is equivalent to the metric on X. Further, from the definition of metric induced by a lamination, each φ_n is a distance dicreasing map. So,

$$\begin{aligned}d_X(\varphi(z_i,t_i),\varphi(z_j,t_j)) &= \lim_{n\to\infty} d_X(\varphi_n(z_i,\psi_n(t_i)),\varphi_n(z_j,\psi_n(t_j))) \\ &= \lim_{n\to\infty}\beta\times d_{\mathcal{L}_n}(\varphi_n(z_i,\psi_n(t_i)),\varphi_n(z_j,\psi_n(t_j))) \\ &\leq \lim_{n\to\infty} d_{\mathbb{D}_\delta}(z_i,z_j) + d_{T_n}(\psi_n(t_i),\psi_n(t_j)) \\ &\leq \lim_{n\to\infty} d_{\mathbb{D}_\delta}(z_i,z_j) + d_{T\setminus\underline{T}}(t_i,t_j) + 2\delta_n \\ &= d_{\mathbb{D}_\delta}(z_i,z_j) + d_{T\setminus\underline{T}}(t_i,t_j)\end{aligned}$$

□

As S is dense in $\mathbb{D}_\delta\times(T\setminus\underline{T})$, given any point in $x\in\mathbb{D}_\delta\times(T\setminus\underline{T})$ there exist a sequence $s_n\in S$ such that $d(s_n,x)$ tends to 0. Define $\varphi(x) = \lim_{n\to\infty}\varphi(s_n)$ where s_n tends to x as n tends to ∞.

The map φ is well defined:

Let $\langle s_n\rangle_n$ and $\langle t_n\rangle_n$ be two sequence which converge to x. Then the sequence $s_1,t_1,s_2,t_2,...$ is Cauchy and so is the sequence $f(s_1),f(t_1),f(s_2),f(t_2),...$ by Lemma 4.3. Thus it converges and has the same limit as $\langle f(s_n)\rangle_n$ and $\langle f(t_n)\rangle_n$. Hence $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} f(t_n)$.

Transition maps behave well

Let x, y be two points in X and $\varphi : \mathbb{D}_\delta \times T \to M, \phi : \mathbb{D}_\delta \times S \to M$ be local parametrisations around x and y created as explained before such that $im(\varphi) \cap im(\phi) \neq \emptyset$. We will prove that $\phi^{-1} \circ \varphi : \varphi^{-1}(\phi(B(0,r) \times T)) \to \phi^{-1}(\varphi(B(0,r) \times S))$ satisfies all conditions embedded L-Lipschitz hyperbolic Riemann surface lamination.

Let $x_n, y_n \in \mathcal{L}_n$ be such that x_n and y_n converge to x and y respectively. Let $\varphi_n : \mathbb{D}_\delta \times T_n \to X$ and $\phi_n : \mathbb{D}_\delta \times S_n \to X$ be cylindrical charts around x_n and y_n respectively. As $im(\varphi) \cap im(\phi) \neq \emptyset$, $im(\varphi_n) \cap im(\phi_n) \neq \emptyset$ for large enough n. Construct $\Psi^n : S \setminus \underline{S} \to S_n$ and $\psi_n : T_n \setminus \underline{T_n} \to T$ as in Theorem 2.14. Then,

$$\begin{aligned}
\phi^{-1} \circ \varphi(z,t) &= \lim_{n\to\infty} (id \times \Psi_n) \circ ((\phi_n)^{-1} \circ \varphi_n) \circ (id \times \psi_n)(z,t) \\
&= \lim_{n\to\infty} (id \times \Psi_n) \circ ((\phi_n)^{-1} \circ \varphi_n)(z, \psi_n(t)) \\
&= \lim_{n\to\infty} (id \times \Psi_n)(\lambda_n(z, \psi_n(t)), \tau_n(\psi_n(t))) \\
&= \lim_{n\to\infty} (\lambda_n(z, \psi_n(t)), \Psi_n(\tau_n(\psi_n(t)))) \\
&= (\lim_{n\to\infty} \lambda_n(z, \psi_n(t)), \lim_{n\to\infty} \Psi_n(\tau_n(\psi_n(t)))).
\end{aligned}$$

It can be proved as in the proof of Proposition 2.15 that given $\lambda_n : \mathbb{D}_\delta \times T_n \to \mathbb{D}_\delta$ and $\tau^n : T_n \to S_n$ there exists $\lambda : \mathbb{D}_\delta \times T \to \mathbb{D}_\delta$ and $\tau : T \to S$ such that $\lim_{n\to\infty} \lambda_n(z, \psi_n(t)) = \lambda(z,t)$ and $\lim_{n\to\infty} \Psi_n(\tau_n(\psi_n(t))) = \tau(t)$. To do this just note that to prove that the function is well defined we only needed that Cauchy sequences map to Cauchy sequences. As Lipschitz maps satisfy this we are done. We have λ for a fixed t is holomorphic by Montel's theorem and Wiestrass's theorem.

Define $\delta'_n = d_\rho(S_n, S)$ and $\delta_n = d_\rho(T_n, T)$. Given $t, s \in T$, choose n so large that $d(s,t) \leq \min\left\{\left(\frac{1}{\delta_n} - 2\delta_n\right), \left(\frac{1}{\delta_n} - 2\delta_n - 2\delta'_n\right)\right\}$. Then,

$$\begin{aligned}
d(\Psi_n(\tau_n(\psi_n(t))), \Psi_n(\tau_n(\psi_n(s)))) &\leq d(\tau_n(\psi_n(t)), \tau_n(\psi_n(s))) + 2\delta'_n (\text{Theorem 2.14}) \\
&\leq L \times d(\psi_n(t), \psi_n(s)) + 2\delta'_n (\tau^n \text{ is } L\text{-Lipshcitz}) \\
&\leq L \times d(t,s) + 2\delta_n + 2\delta'_n (\text{Theorem 2.14}).
\end{aligned}$$

As $2\delta_n + 2\delta'_n$ tends to zero as n tends to infinity, this shows that τ is L- Lipschitz.

Thus, $\mu \in \mathcal{M}(X, L, M, \delta)$. As the sequence μ_n was arbitrary, $\mathcal{M}(X, L, M, \delta)$ is closed subset of the compact set $(\mathcal{B}(X), d_\pi)$ and hence compact. □

5 Proof Of Theorem 1.3

Let $\mathcal{L}_0$ be an embedded closed J-holomorphic curve in X (Recall that such J-holomorphic curves exist. See Section 3.1 in [MS12]). We view it as an embedded measured Riemann surface lamination in X. We know that $\mathcal{L}_0$ induces a Borel measure μ satisfying properties P-1 and P-2. Consider the sequence $\{\mu_i = \varphi_*(\mu_{i-1})\}$, where $\varphi_*(\mu)$ denotes the push forward measure defined as

$$\varphi_*(\mu)(B) = \mu(\varphi^{-1}(B))$$

for any Borel set B. Then, $\mu_i \in \mathcal{M}(X)$ for all i. By Theorem 1.2 there is a subsequence which converge to measure $\overline{\mu} \in \mathcal{M}(X)$. Therefore, there is an embedded measured Riemann surface lamination, denote it by $\overline{\mathcal{L}}$, which induces $\overline{\mu}$.

Lemma 5.1. *$\varphi(Supp(\overline{\mu})) \subset Supp(\overline{\mu})$, that is, φ fixes the support of $\overline{\mu}$ (as a set). Hence φ^n fixes $Supp(\overline{\mu})$.*

Proof. Assume there is a point $p \in \mathrm{Supp}(\overline{\mu})$ such that $\varphi(p) \notin \mathrm{Supp}(\overline{\mu})$. As $\mathrm{Supp}(\overline{\mu})$ is compact and satifies the property P-2, there is a cylindrical chart ψ at $\varphi(P)$ (for $\varphi(\overline{\mathcal{L}})$) such that $Im(\psi) \cap \overline{\mathcal{L}} = \emptyset$. Thus, $\overline{\mu}(Im(\psi))$ is zero by definition. As, $\varphi_*(\overline{\mu}) = \overline{\mu}$, $\varphi_*(\overline{\mu})(Im(\psi))$ is zero. Thus, $\overline{\mu}(\varphi^{-1}(Im(\psi))) = \overline{\mu}(Im(\varphi^{-1} \circ \psi))$ is zero. But, $p \in \overline{\mathcal{L}}$, $p \in Supp(\overline{\mu})$ and $Im(\varphi^{-1} \circ \psi))$ is an open set containing p. So, $\overline{\mu}(Im(\varphi^{-1} \circ \psi))$ cannot be zero, a contradiction. So, $\varphi(p) \in \mathrm{Supp}(\overline{\mu})$ for all points $p \in \mathrm{Supp}(\overline{\mu})$. □

Let $\{(U_i, \psi_i)\}_{i \in I}$ be an atlas for the lamination $\overline{\mathcal{L}}$. Then, the collection $\{\phi^n(U_i), \phi^{-n} \circ \psi_i\}_{i \in I}$ forms an atlas for $\phi^n(\overline{\mathcal{L}})$. The two laminations are same if and only if, for each j, the chart $(\phi^n(U_j), \phi^{-n} \circ \psi_j)$ is compatible with the atlas $\{(U_i, \psi_i)\}_{i \in I}$.

Now, there are two cases. Let us first assume that the derived lamination (lamination minus the isolated leaves) of $\overline{\mathcal{L}}$ is empty, i.e., $\overline{\mathcal{L}}$ consists of only isolated leaves. By Lemma 5.1, we have $\varphi_*\overline{\mu} = \overline{\mu}$. This implies that φ permutes the leaves of $\overline{\mathcal{L}}$. Hence ϕ fixes $\overline{\mathcal{L}}$. Moreover, some power φ^k fixes every leaf (as a set) of $\overline{\mathcal{L}}$.

Now, let us assume that the derived lamination $\overline{\mathcal{L}}'$ of $\overline{\mathcal{L}}$ is not empty and no power φ^k with $k \geq 2$ fixes the lamination (J-holomorphic curve) $\mathcal{L}_0$.' In particular, this implies that $\varphi^p(\mathcal{L}_0)$ and $\varphi^q(\mathcal{L}_0)$ are distinct J-holomorphic curves for $p \neq q$ positive integers. Recall that the subsequence $\{\varphi^{n_k}(\mathcal{L})\}$ converges to $\overline{\mathcal{L}}$.

For any fixed positive integer r, observe that $\mu_{n_k}(\varphi^{n_k}(\mathcal{L}_0) \cap \varphi^{n_k+r}(\mathcal{L}_0)) = \mu_{n_k+r}(\varphi^{n_k}(\mathcal{L}_0) \cap \varphi^{n_k+r}(\mathcal{L}_0)) = 0$ as distinct compact J-holomorphic curves intersect in at most finitely many points. Now, we claim that $\overline{\mu}(\overline{\mathcal{L}} \pitchfork \varphi^r(\overline{\mathcal{L}}')) = 0$. (recall that $\pitchfork$ denotes the transverse intersection.) Assume contrary that $\overline{\mu}(\overline{\mathcal{L}} \pitchfork \varphi^r(\overline{\mathcal{L}}')) > 0$. Then, for large enough n_k, we must have $\mu_{n_k}(\varphi^{n_k}(\mathcal{L}_0) \cap \varphi^{n_k+r}(\mathcal{L}_0)) > 0$. This contradicts our earlier observation. Thus, we have $\overline{\mu}(\overline{\mathcal{L}} \pitchfork \varphi^r(\overline{\mathcal{L}}')) = 0$. Now, notice that $\overline{\mathcal{L}} \pitchfork \varphi^r(\overline{\mathcal{L}}') = \emptyset$. If $x \in \overline{\mathcal{L}} \pitchfork \varphi^r(\overline{\mathcal{L}}')$ then consider a chart (U, ψ) around the point x for the lamination $\overline{\mathcal{L}}$ such that all the plaques in (U, ψ) intersect the leaves of the lamination $\varphi^r(\overline{\mathcal{L}}')$ transversely. We see that $\overline{\mu}(U) > 0$ as U is an open subset of $\mathrm{Supp}(\overline{\mu})$ (in the subspace topology). Recall, by Lemma 5.1 $\varphi^r_*(\overline{\mu}) = \overline{\mu}$. Therefore, $\varphi^r(\overline{\mathcal{L}})$ also induces the measure $\overline{\mu}$. Hence $\overline{\mu}(U) = \overline{\mu}(U \cap \varphi^r(\overline{\mathcal{L}}')) > 0$. Therefore $0 < \overline{\mu}(U \cap \varphi^r(\overline{\mathcal{L}}')) \leq \overline{\mu}(\overline{\mathcal{L}} \pitchfork \varphi^r(\overline{\mathcal{L}}'))$. A contradiction.

This implies that $\varphi^r(\overline{\mathcal{L}}')$ and $\varphi^s(\overline{\mathcal{L}}')$ are disjoint whenever $r \neq s$. We take the union $T = \cup \varphi^r(\overline{\mathcal{L}}')$. In the following lemma, we prove that the closure $\overline{T}$ is also a Riemann surface lamination. Thus it follows that $\phi(\overline{T}) = \overline{T}$.

Lemma 5.2. *Let $T = \sqcup \mathcal{L}_k$ denote the disjoint union of embedded Riemann surface laminations $\mathcal{L}_k$ in X. Then the closure $\overline{T}$ is also an embedded Riemann surface lamination X.*

Proof. We take the union of all leaves (Riemann surfaces in this case) of all laminations $\mathcal{L}_k$. Then $T = \sqcup L_j$. Let $x \in \overline{T}$. There is a sequence of points $x_k \in \mathcal{L}_k$ such that $x_k \to x$.

Let L_k denote the leaf which contains x_k. Thus, we have a sequence of disjoint leaves L_k. We view this sequence as a sequence of embedded Riemann surface laminations. By the compactness theorem 1.2, we get a subsequence L_{k_j} that converges to an embedded Riemann surface lamination $\tilde{L}$. Clearly $x \in \tilde{L}$. Further, any point $y \in \tilde{L}$ is a point in $\overline{T}$, therefore $\tilde{L} \subset \overline{T}$.

Observe that $\tilde{L} \cap L_n = \emptyset$ for any leaf L_n in T. To this end, we note that if $\tilde{L} \cap L_n \neq \emptyset$ then $L_n \cap L_{k_j} \neq \emptyset$ for large enough j for the above subsequence $\{L_{k_j}\}$ as it converges to $\tilde{L}$. This contradicts the assumption that all the leaves in T are disjoint.

Now, suppose that there are two sequences of leaves in T, $\{L_j\}$ and $\{L_k\}$ converging to two distinct laminations $\tilde{L}_0$ and $\tilde{L}_1$ respectively. Then $\tilde{L}_0 \cap \tilde{L}_1 = \emptyset$. To see this, assume contrary that $\tilde{L}_0 \cap \tilde{L}_1 \neq \emptyset$. Since $\tilde{L}_0$ and $\tilde{L}_1$ are distinct laminations, it follows that $\tilde{L}_0 \pitchfork \tilde{L}_1 \neq \emptyset$ (i.e, some leaves intersect transversely). Then for large enough j and m, we have $L_j \cap L_m \neq \emptyset$. This is contradiction to our assumption that all L_j's are disjoint leaves. Thus, $\overline{T}$ is a disjoint union of embedded Riemann surface laminations. Hence, $\overline{T}$ is an embedded Riemann surface lamination.

□

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